Intermediate Mathematical Challenge

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Solutions and investigations

February 2022

These solutions augment the shorter solutions also available online. The shorter solutions in many cases omit details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to challenges@ukmt.org.uk.

The Intermediate Mathematical Challenge (IMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that sometimes you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the IMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with each step explained (or, occasionally, left as an exercise). We therefore hope that these solutions can be used as a model for the type of written solution that is expected when a complete solution to a mathematical problem is required (for example, in the Intermediate Mathematical Olympiad and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
B	A	B	D	C	D	E	A	E	C	E	A	C	D	E	D	C	B	A	D	B	D	E	B	C

1. How many hours is 6 minutes?

A 0.06 B 0.1 C 0.6 D 10 E 360

SOLUTION **B**

There are 60 minutes in each hour. Therefore 6 minutes, as a fraction of an hour, is $\frac{6}{60}$ which is equal to $\frac{1}{10}$. As a decimal $\frac{1}{10}$ is 0.1.

FOR INVESTIGATION

1.1 How many hours is 27 minutes expressed as a decimal?

1.2 How many minutes are there in 0.35 hours?

1.3 How many minutes are there in 0.47 hours?

2. My recipe for apple crumble uses 100 g of flour, 50 g of butter and 50 g of sugar to make the crumble topping. When my family come for a meal, I have to use two and a half times each amount to provide enough crumble. In total, how much crumble topping do I then make?

A 0.5 kg B 2 kg C 2.5 kg D 5 kg E 50 kg

SOLUTION **A**

The recipe makes $100\,\text{g} + 50\,\text{g} + 50\,\text{g} = 200\,\text{g}$ of topping. When I multiply these amounts by two and a half the amount of topping that I make is

$$2\tfrac{1}{2} \times 200\,\text{g} = \frac{5}{2} \times 200\,\text{g} = 500\,\text{g} = 0.5\,\text{kg}.$$

FOR INVESTIGATION

2.1 Philip Harben's recipe for apple crumble uses 100 g of flour, 50 g of butter, 25 g of sugar, 225 g of apples, and nothing else. I find that I have to use three and a half times these amounts when my family comes round for a meal.

What is the total weight of the apple crumble that I make, using Philip Harben's recipe, when the family comes for a meal?

3. In the Caribbean, loggerhead turtles lay three million eggs in twenty thousand nests. On average, how many eggs are in each nest?

A 15 B 150 C 1500 D 15 000 E 150 000

Solution **B**

The average number of eggs in each nest is

$$\frac{3\,000\,000}{20\,000} = \frac{300}{2} = 150.$$

For investigation

3.1 Asian giant soft shell turtles lay an average of fifty eggs in each nest.

How many eggs do they lay in twenty-five thousand nests?

4. Workers digging a tunnel for an underground railway complete 5 metres of tunnel on a typical day.
Working every day, how long will it take them to dig a tunnel of length 2 kilometres?

A three months B six months C just under a year
D just over a year E nearly two years

Solution **D**

Since 2 kilometres is 2000 metres, and 5 metres of tunnel is dug on a typical day, it will take around $\frac{2000}{5} = 400$ days.

There are 365 days in a year, or 366 in a leap year.

Therefore 400 days is just over a year.

For investigation

4.1 The workers are offered a substantial bonus if they complete digging the tunnel within 10 months. To the nearest metre, how many metres of tunnel, on average, will they need to dig each day to obtain the bonus?

5. Which of the following has the same value as $10\,006 - 8008$?

A $10\,007 - 8007$ B $100\,060 - 80\,080$ C $10\,000 - 8002$
D $106 - 88$ E $5003 - 4004$

SOLUTION **C**

COMMENTARY

One method would be to do a lot of arithmetic to calculate the value of $10\,006 - 8008$, and then the values of the options until we find one which has the same value.

However, it is sensible to look for something better. You might then notice that $10\,000$ is 6 less than $10\,006$ and also 8002 is 6 less than 8008.

This leads to the following quick solution.

We have,

$$\begin{aligned} 10\,006 - 8008 &= (10\,000 + 6) - (8002 + 6) \\ &= 10\,000 + 6 - 8002 - 6 \\ &= 10\,000 - 8002. \end{aligned}$$

Therefore the correct option is $10\,000 - 8002$.

FOR INVESTIGATION

5.1 Which of the following has the same value as $100\,012 - 90\,016$?

(a) $100\,022 - 90\,006$,

(b) $200\,024 - 180\,032$,

(c) $100\,007 - 90\,011$,

(d) $102\,012 - 90\,216$.

5.2 For which value of x does $54\,321 - x$ have the same value as $98\,765 - 66\,666$?

5.3 For which value of x does $1234 \times x$ have the same value as 2468×8642?

6. What is 20% of $3\frac{3}{4}$?

A $\dfrac{123}{200}$ B $\dfrac{13}{20}$ C $\dfrac{7}{10}$ D $\dfrac{3}{4}$ E $\dfrac{4}{5}$

SOLUTION **D**

We have

$$20\% = \frac{1}{5},$$

and

$$\begin{aligned} 3\frac{3}{4} &= \frac{12}{4} + \frac{3}{4} \\ &= \frac{15}{4}. \end{aligned}$$

Therefore

$$\begin{aligned} 20\% \text{ of } 3\tfrac{3}{4} &= \frac{1}{5} \times \frac{15}{4} \\ &= \frac{3}{4}. \end{aligned}$$

FOR INVESTIGATION

6.1 Find the values of

(a) 25% of $2\frac{2}{7}$,

(b) $33\frac{1}{3}$% of $4\frac{1}{8}$,

(c) $17\frac{1}{2}$% of 360.

6.2 (a) What percentage of 40 is 32?

(b) What percentage of 60 is 32?

(c) What percentage of 60 is $4\frac{4}{5}$?

(d) What percentage of $11\frac{1}{4}$ is $1\frac{4}{5}$?

7. A function machine does the four operations shown in order.

$$\longrightarrow \boxed{\div 3} \longrightarrow \boxed{-10} \longrightarrow \boxed{\div 3} \longrightarrow \boxed{-10} \longrightarrow$$

Iris inputs a positive integer and the output is also a positive integer.
What is the smallest possible number which Iris could have input?

A 9 B 84 C 102 D 120 E 129

Solution **E**

Commentary

We give three methods for answering this question.

The first is the most straightforward: try each of the options in turn. However this method involves quite a lot of work. Also, of course, it relies on the assumption that one of the options is correct. Hence it does not produce a completely justified mathematical solution.

The second method involves working backwards from a guess as to what the answer might be. Fortunately, the guess is correct. So this method is efficient.

The third method uses algebra and yields a complete mathematical solution.

Method 1

We input each of the options in turn and see what output results. For example, with input 84, the calculation

$$84 \longrightarrow \boxed{\div 3} \xrightarrow{28} \boxed{-10} \xrightarrow{18} \boxed{\div 3} \xrightarrow{6} \boxed{-10} \longrightarrow -4$$

shows that the output is -4. This is not a positive integer so option B is not correct.

We leave checking the other options to Problem 7.1. It will be seen that, of the given options, only 129 produces an output that is a positive integer.

Hence, assuming that one of the options gives the correct answer, we deduce that 129 is the smallest possible number that Iris could have input.

Method 2

The smallest positive integer is 1. By reversing the function machine we check whether using any of the options as input gives 1 as the output.

Note that when we reverse the function machine we need to replace each operation with its inverse. Thus we replace $\div 3$ by $\times 3$ and -10 by $+10$:

$$129 \longleftarrow \boxed{\times 3} \xleftarrow{43} \boxed{+10} \xleftarrow{33} \boxed{\times 3} \xleftarrow{11} \boxed{+10} \longleftarrow 1$$

We see from this that the output 1 corresponds to the input 129. We deduce that 129 is the smallest integer that Iris could have input.

Note: This assumes that if we input a number that is smaller than 129 the output will be smaller than 1. See Problem 7.3 for how to justify this assumption.

Method 3

We see that if we input x the output will be $\frac{1}{9}x - \frac{40}{3}$ (you are asked to check this in Problem 7.2):

$$x \longrightarrow \boxed{\div 3} \xrightarrow{\frac{1}{3}x} \boxed{-10} \xrightarrow{\frac{1}{3}x - 10} \boxed{\div 3} \xrightarrow{\frac{1}{9}x - \frac{10}{3}} \boxed{-10} \longrightarrow \frac{1}{9}x - \frac{40}{3}$$

We now put

$$\frac{1}{9}x - \frac{40}{3} = n.$$

If we multiply both sides of this equation by 9, we obtain

$$x - 120 = 9n.$$

This gives

$$x = 120 + 9n.$$

From this last equation we see that the larger the value of n, the larger the value of x. So to get the smallest possible input, we need n to be as small as possible.

Because n is a positive integer its smallest possible value is given by $n = 1$. From the final equation above, we see that for $n = 1$ we have $x = 120 + 9 = 129$.

Therefore the smallest number Iris could have input is 129.

For investigation

7.1 Calculate the output from the function machine for each of the inputs 9, 102, 120, and 129.

7.2 Check that $\left(\frac{1}{9}x - \frac{10}{3}\right) - 10 = \frac{1}{9}x - \frac{40}{3}$.

7.3 In Method 3 we have shown that for input x the output is $\frac{1}{9}x - \frac{40}{3}$. Use this to show that the smaller the input, the smaller the output will be. In other words, show that if $x < y$, then $\frac{1}{9}x - \frac{40}{3} < \frac{1}{9}y - \frac{40}{3}$.

7.4 What is the smallest possible positive integer that Iris can input to the function machine

$$\longrightarrow \boxed{\times 7} \longrightarrow \boxed{-9} \longrightarrow \boxed{\div 3} \longrightarrow \boxed{-2} \longrightarrow$$

to obtain an output that is a positive integer?

8. What is the difference between 40% of 50% of 60 and 50% of 60% of 70?

A 9 B 8 C 7 D 6 E 5

SOLUTION **A**

40% of 50% of 60 is $0.4 \times 0.5 \times 60 = 12$.

50% of 60% of 70 is $0.5 \times 0.6 \times 70 = 21$.

Therefore the difference is $21 - 12 = 9$.

FOR INVESTIGATION

8.1 What is the difference between 20% of 30% of 40 and 60% of 70% of 80?

8.2 What is the difference between 15% of 25% of 80 and 25% of 35% of 40?

9. A number x is greater than 2022. Which is the smallest of the following?

A $\dfrac{x}{2022}$ B $\dfrac{2022}{x-1}$ C $\dfrac{x+1}{2022}$ D $\dfrac{2022}{x}$ E $\dfrac{2022}{x+1}$

Solution **E**

Note: Strictly speaking, the question should have specified that x is an *integer*. We use this to argue that because $x > 2022$, it follows that $x - 1 \geq 2022$.

Because $x > 2022$, we have $2022 < x < x + 1$ and therefore

$$1 < \frac{x}{2022} < \frac{x+1}{2022}.$$

Also $x + 1 > x > x - 1 \geq 2022$ and therefore

$$\frac{2022}{x+1} < \frac{2022}{x} < \frac{2022}{x-1} \leq 1.$$

It follows that

$$\frac{2022}{x+1} < \frac{2022}{x} < \frac{2022}{x-1} < \frac{x}{2022} < \frac{x+1}{2002}.$$

It follows that of the given options it is $\dfrac{2022}{x+1}$ that is the smallest.

For investigation

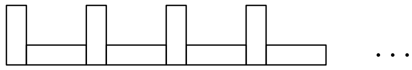
9.1 The integer n is greater than 1. Which of the following is the largest?

(a) $\dfrac{1000}{n^2}$, (b) $\dfrac{1000}{n^2+1}$, (c) $\dfrac{1000}{(n-1)(n+1)}$.

9.2 The integer n is greater than 1. Which of the following is the largest?

(a) $\dfrac{n^2+1000}{n^2}$, (b) $\dfrac{n^2+1000}{n^2+1}$, (c) $\dfrac{n^2+1000}{(n-1)(n+1)}$.

10. One hundred rectangles are arranged edge-to-edge in a continuation of the pattern shown.

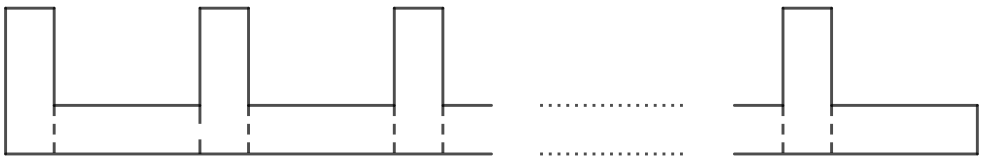


Each rectangle measures 3 cm by 1 cm. What is the perimeter, in cm, of the completed shape?

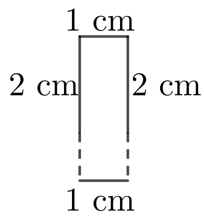
A 800 B 700 C 602 D 600 E 502

Solution **C**

In the pattern there are 50 vertical rectangles and 50 horizontal rectangles as shown in the diagram.

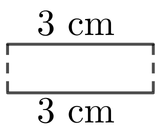


In this diagram the perimeter is shown by a solid line. Edges, or parts of edges, of the rectangles that do not form part of the perimeter are indicated by broken lines.



Each vertical rectangle, except for the rectangle at the left hand end, contributes 1 cm + 2 cm + 1 cm + 2 cm = 6 cm to the length of the perimeter. The left hand rectangle contributes an additional 1 cm.

Therefore the total amount that the 50 vertical rectangles contribute to the length of the perimeter is 50×6 cm + 1 cm = 301 cm.



Each horizontal rectangle, except for the rectangle at the right hand end, contributes 3 cm + 3 cm = 6 cm to the length of the perimeter. The right hand rectangle contributes an additional 1 cm.

Therefore the total amount that the 50 horizontal rectangles contribute to the length of the perimeter is 50×6 cm + 1 cm = 301 cm.

It follows that the total length of the perimeter is 301 cm + 301 cm = 602 cm.

For investigation

10.1 Two hundred rectangles, each measuring 3 cm by 1 cm, are used to make a pattern similar to that shown in this question.

What is the length of the perimeter of the completed shape?

10.2 $2n$ rectangles, each measuring 3 cm by 1 cm, are used to make a pattern similar to that shown in this question.

What is the length of the perimeter of the completed shape?

10.3 $2n$ rectangles, each measuring a cm by b cm, with $a > b$, are used to make a pattern similar to that shown in this question.

What is the length of the perimeter of the completed shape?

11. The Universal Magazine of Knowledge and Pleasure (Vol. 1, 1747) asked the following question.
"What number is that, whose quarter shall be 9 more than the whole?"
What is the correct answer?

A 12 B 9 C 8 D -8 E -12

SOLUTION **E**

COMMENTARY

The most straightforward method is to subtract each of the numbers given as options from one quarter of the number, and check which calculation yields the answer 9. This is our first method. It involves quite a lot of arithmetic and only produces the answer because one of the options is correct.

Our second method uses algebra, and is more efficient.

METHOD 1

We have

$$\begin{aligned} &\tfrac{1}{4}(12) - 12 = 3 - 12 = -9 \neq 9,\\ &\tfrac{1}{4}(9) - 9 = \frac{9}{4} - \frac{36}{4} = \frac{-27}{4} \neq 9,\\ &\tfrac{1}{4}(8) - 8 = 2 - 8 = -6 \neq 9,\\ &\tfrac{1}{4}(-8) - (-8) = -2 + 8 = 6 \neq 9,\\ \text{and}\quad &\tfrac{1}{4}(-12) - (-12) = -3 + 12 = 9. \end{aligned}$$

Therefore the correct answer is -12.

METHOD 2

Let x be the number we seek.

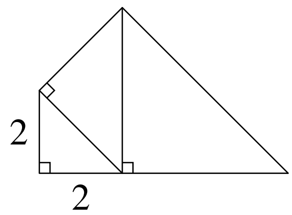
Then $\frac{1}{4}x - x = 9$. Therefore $-\frac{3}{4}x = 9$. Hence $x = -\frac{4}{3} \times 9 = -12$.

FOR INVESTIGATION

11.1 Which number is 8 less than one fifth of the number?

12. The shape shown is made up of three similar right-angled triangles.
The smallest triangle has two sides of side-length 2, as shown.

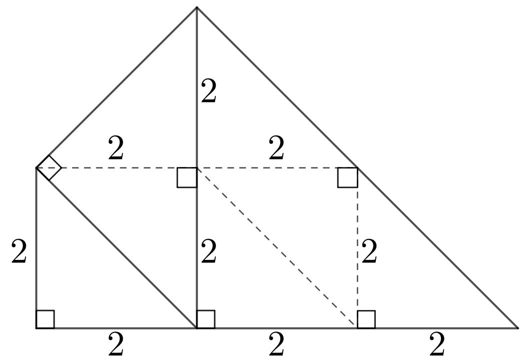
What is the area of the shape?



A 14 B $12 + 12\sqrt{2}$ C 28 D $24 + 20\sqrt{2}$ E 56

Solution **A**

The diagram on the right shows that the shape may be divided into seven triangles each congruent to the smallest triangle in the diagram given in the question.



Using the formula $\frac{1}{2}$(base × height) for the area of a triangle, we see that the area of each of the 7 congruent triangles is $\frac{1}{2}(2 \times 2) = 2$.

Therefore the area of the shape is $7 \times 2 = 14$.

For investigation

12.1 Prove that the seven triangles used in the above solution are congruent.

12.2 (a) Find the side lengths of all three triangles shown in the diagram given in the question.

(b) Hence find the areas of each of these triangles and verify that the total area of the shape is 14.

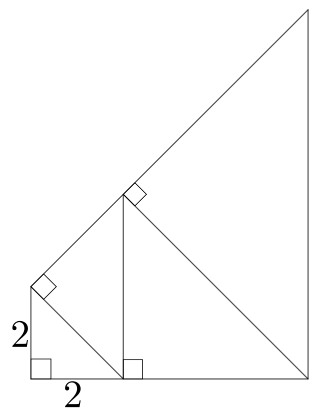
12.3 (a) Find the lengths of the hypotenuses of the three triangles in the diagram of the question.

(b) Use the fact that the areas of similar triangles are in proportion to the squares of the lengths of corresponding sides, to deduce that the areas of the three triangles in the diagram of the question are in the ratio $1 : 2 : 4$. Deduce that the area of the whole shape is 7 times the area of the smallest triangle.

12.4

(a) Use the method of Problem 12.3 to find the area of the shape in the diagram on the right which is obtained from the diagram in the question by adding a fourth similar right-angled triangle.

(b) Find a formula in terms of n for the area of the shape made up in the same way from n similar right-angled triangles.



13. How many sets of three consecutive integers are there in which the sum of the three integers equals their product ?

A 0 B 2 C 3 D 4 E 5

SOLUTION **C**

> COMMENTARY
>
> Your first thought here might be to take n, $n+1$ and $n+2$ to be the three consecutive integers we are seeking. This would not be wrong, but the choice of $n-1$, n and $n+1$ makes the algebra a little easier.

We let $n-1$, n and $n+1$ be three consecutive integers whose sum is equal to their product.

Then we have

$$(n-1)+n+(n+1)=(n-1)n(n+1).$$

This gives

$$3n=n(n^2-1).$$

This equation may be rearranged to give

$$n(n^2-1)-3n=0,$$

that is,

$$n(n^2-1-3)=0.$$

This gives

$$n(n^2-4)=0.$$

The left hand side of this equation may be factorized to give

$$n(n-2)(n+2)=0.$$

Therefore

$$n=0 \text{ or } n=2 \text{ or } n=-2.$$

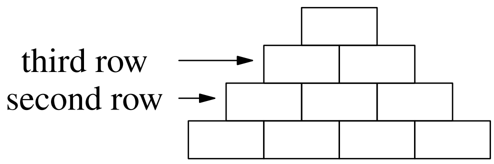
It follows that there are three sets of consecutive integers whose sum is equal to their product, namely, $\{-1,0,1\}$, $\{-3,-2,-1\}$ and $\{1,2,3\}$.

FOR INVESTIGATION

13.1 How many sets of three different positive integers are there in which the sum of the three integers equals their product?

13.2 Show that there are infinitely many sets of three different integers in which the sum of the three integers equals their product.

14. In a number pyramid, each cell above the bottom row contains the sum of the numbers in the two cells immediately below it. The three numbers on the second row are all equal, and are all integers. Which of these statements *must* be true?

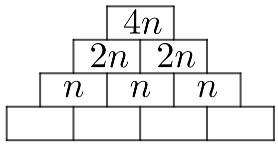


A The bottom row contains at least one zero
B The third row contains at least one zero
C The top number is a multiple of three
D The top number is a multiple of four
E None of the above

SOLUTION **D**

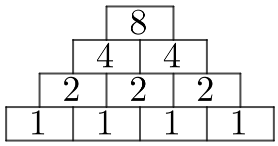
Let n be the integer that is in all of the cells in the second row.

Because the number in each cell is the sum of the numbers in the two cells immediately below it, in the third row the number in each cell is $2n$ and the top number is $4n$.



Because $4n$ is always a multiple of 4, in the context of the IMC this is enough to show that the correct option is D.

For a complete solution it is necessary to show that none of the other options is correct. In the example on the right, all the numbers in the second row are equal, but none of A, B or C is true. So these options are not correct.



Finally, because option D is correct, it follows that option E is also not correct.

FOR INVESTIGATION

14.1 Give an example to show that it possible for the numbers in the bottom row to not be all equal, while the numbers in second row are all equal.

14.2 Is it possible for the four numbers in the bottom row to be all different, while the numbers in the second row are all equal?

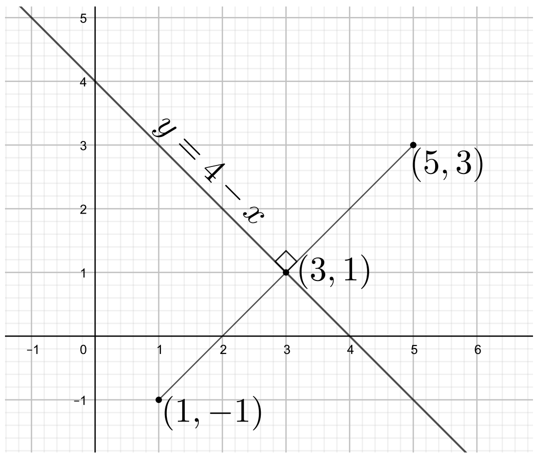
15. Reflection in the line l transforms the point with coordinates $(5, 3)$ into the point with coordinates $(1, -1)$.
What is the equation of the line l?

A $y = x - 2$ B $y = 1$ C $x = 3$ D $y = 2 - x$ E $y = 4 - x$

Solution **E**

The required line l goes through the midpoint of the line segment joining the points with coordinates $(5, 3)$ and $(1, -1)$ and is perpendicular to it. In other words, l is the *perpendicular bisector* of this line segment.

The midpoint of this line segment has coordinates $\left(\frac{1}{2}(5 + 1), \frac{1}{2}(3 + (-1))\right)$, that is, $(3, 1)$.



The slope of the line segment is

$$\frac{3 - (-1)}{5 - 1} = \frac{4}{4} = 1.$$

Therefore the line perpendicular to the line segment has slope -1.

Thus the line l is the line with slope -1 which goes through the point $(3, 1)$. Hence the equation of l is

$$y - 1 = -1(x - 3),$$

that is

$$y - 1 = -x + 3.$$

This last equation may be arranged as $y = 4 - x$.

For investigation

15.1 Find the equation of the line which has the property that reflection in the line transforms the point with coordinates $(1, 2)$ into the point with coordinates $(3, 6)$.

16. What is half of 4^{2022}?

A 4^{1011} B 2^{4044} C 4^{2021} D 2^{4043} E 2^{1011}

SOLUTION **D**

We have $4^{2022} = (2^2)^{2022} = 2^{4044}$

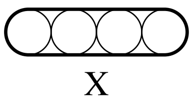
Therefore half of 4^{2022} is $\dfrac{2^{4044}}{2} = 2^{4043}$.

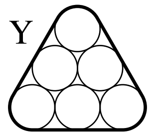
FOR INVESTIGATION

16.1 Express one third of 9^{1000} in the form 3^n, where n is a positive integer.

16.2 Express $8^{1000} \div 1024$ in the form 2^n, where n is a positive integer.

17. The first figure shows four touching circles of radius 1 cm in a horizontal row, held together tightly by an outer band X.

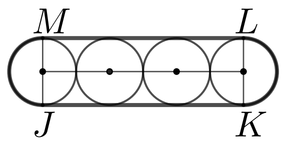




The second figure shows six touching circles of radius 1 cm, again held tightly together by a surrounding band Y.
Which of the following statements is true?

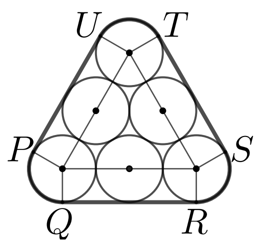
A X is 2 cm longer than Y
B X is 1 cm longer than Y
C X and Y are the same length
D Y is 1 cm longer than X
E Y is 2 cm longer than X

SOLUTION **C**



We see from the diagram on the right that the band X consists of two straight line sections, JK and ML each of length 6 cm and two semi-circular arcs, KL and MJ, which together make up a circle of radius 1 cm and hence with circumference 2π cm.

Therefore the length of X is $(2 \times 6 + 2\pi)$ cm, that is $(12 + 2\pi)$ cm.



Similarly, from the diagram on the right we see that the band Y consists of three straight line sections, QR, ST and UP each of length 4 cm and three equal arcs, PQ, RS and TU which together make up a circle of radius 1 cm.

Therefore the length of Y is $(3 \times 4 + 2\pi)$ cm, that is $(12 + 2\pi)$ cm.

Therefore X and Y are the same length.

FOR INVESTIGATION

17.1 Explain why the three arcs, PQ, RS and TU each make up one third of the circle of which they are part.

18. Dick Turnip sold his horse, Slack Bess, for £56. The percentage profit he made was numerically the same as the cost, in pounds, of his horse. What was the cost of his horse?

A £36 B £40 C £45 D £48 E £50

Solution **B**

Commentary

This is a case where trying the options one by one turns out to be much easier than using algebra. Our second method uses the algebraic approach to show how a question of this type could be solved in the absence of options one of which we expect to be correct.

Method 1

We work out the percentage profit for each of the options in turn until we find the one where this is the same as the cost, in pounds, of the horse.

Suppose that the horse cost £36. Then the profit was £56 − £36 = £20. Now £20 as a percentage of the cost price £36 is

$$\frac{20}{36} \times 100 = \frac{500}{9} = 55\tfrac{5}{9} \neq 36.$$

Therefore £36 is not the correct answer.

Suppose that the horse cost £40. Then the profit was £56 − £40 = £16. Now £16 as a percentage of the cost price £40 is

$$\frac{16}{40} \times 100 = 40.$$

Therefore £40 is the correct answer.

Method 2

Let £C be the cost price of the horse. Then the profit was £56 − £C = £$(56 - C)$. As a percentage of the cost price this profit is

$$\frac{56 - C}{C} \times 100.$$

We therefore seek the value of C for which

$$\frac{56 - C}{C} \times 100 = C.$$

This equation is equivalent to

$$100(56 - C) = C^2.$$

This last equation may be rearranged to give

$$C^2 + 100C - 5600 = 0,$$

which is equivalent to

$$(C - 40)(C + 140) = 0.$$

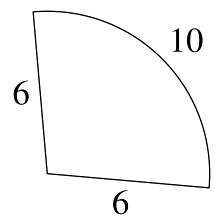
It follows that either $C = 40$ or $C = -140$. The cost price must be positive. We therefore conclude that the horse cost £40.

For investigation

18.1 Calculate the percentage profit corresponding to the cost prices given in options C, D and E, that is £45, £48 and £50.

18.2 Harry Parsnip sold his horse for £96. His percentage profit was numerically the same as the cost, in pounds, of the horse. What was the cost of the horse?

19. A sector of a circle has radius 6 and arc length 10 , as shown.



What is the area of the sector?

A 30 B 36 C 40 D 60 E 66

Solution **A**

Let the area of the sector be X.

Using the formula πr^2 for the area of a circle with radius r, we deduce that the circle has area $\pi 6^2$, that is, 36π.

From the formula $2\pi r$ for the circumference of a circle with radius r, we conclude that the circumference of the circle has length $2\pi 6$, that is, 12π.

The ratio of the area of the sector of a circle to the area of the circle is the same as the ratio of the arc length of the sector to the length of the circumference.

Therefore

$$\frac{X}{36\pi} = \frac{10}{12\pi}.$$

It follows that

$$X = \frac{10}{12\pi} \times 36\pi = \frac{360\pi}{12\pi} = 30.$$

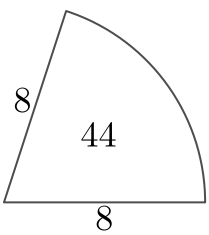
For investigation

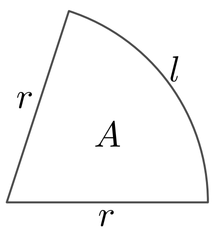
19.1

(a) A sector of a circle with radius 8 has area 44. What is the arc length of the sector?

(b) A sector of a circle with radius r has arc length l and area A.

Find an equation which relates r, l and A.





20. Aroon is asked to choose five integers so that the mode is 2 more than the median and the mean is 2 less than the median. What is the largest possible value of the range of Aroon's five integers?

A 2 B 5 C 12 D 15
E The largest possible range depends on the integers chosen

SOLUTION **D**

Let p, q, r, s and t be five integers whose mode is 2 more than the median, and whose mean is 2 less than the median.

We suppose that $p \leq q \leq r \leq s \leq t$. Then the median of these integers is r. Hence the mode is $r + 2$ and the mean is $r - 2$.

For the mode to be $r + 2$, at least two of the five integers need to be equal to $r + 2$. Since $p \leq q \leq r$, it must be that $s = t = r + 2$. We also note that for $r + 2$ to be the mode no other value can be repeated. Hence $p < q < r$.

It follows that the mean of the five integers is

$$\frac{p + q + r + (r + 2) + (r + 2)}{5} = \frac{p + q + 3r + 4}{5}.$$

Therefore

$$\frac{p + q + 3r + 4}{5} = r - 2.$$

Hence

$$p + q + 3r + 4 = 5r - 10$$

and so

$$p + q = 2r - 14.$$

The range is $(r + 2) - p$. For this to be as large as possible we need p to be as small as possible, and hence q to be as large as possible.

Because q is an integer and $q < r$, the largest possible value of q is given by $q = r - 1$.

Then, from the equation $p + q = 2r - 14$, we have $p = 2r - 14 - q = 2r - 14 - (r - 1) = r - 13$.

We have thus shown that, for the largest possible range, the five integers will be

$$r - 13,\ r - 1,\ r,\ r + 2,\ r + 2.$$

The range of these integers is $(r + 2) - (r - 13) = 15$.

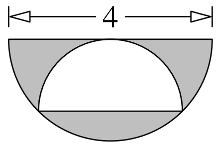
FOR INVESTIGATION

20.1 What is the largest possible range of a set of five integers whose mode is 3 more than its median, and whose mean is 3 less than its median?

20.2 What is the largest possible range of a set of five integers whose mode is k more than its median, and whose mean is k less than its median?

21. The diagram shows a shaded semicircle of diameter 4, from which a smaller semicircle has been removed. The two semicircles touch at exactly three points.

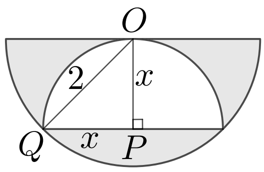
What fraction of the larger semicircle is shaded?



A $\frac{2}{\pi}$ B $\frac{1}{2}$ C $\frac{\sqrt{2}}{3}$ D $\frac{\sqrt{2}}{2}$ E $\frac{3}{4\pi}$

Solution **B**

We let O be the centre of the larger semicircle, P be the centre of the smaller semicircle, and Q be one of the points, other than O, where the two semicircles meet.



We let x be the radius of the smaller semicircle.

We see that OPQ is a right-angled triangle in which $OP = QP = x$. (You are asked to prove these facts in Problem 21.1.)

Since OQ is a radius of the larger semicircle, it has length 2.

By Pythagoras' Theorem applied to the triangle OPQ, we have $x^2 + x^2 = 2^2$. Therefore $x^2 = 2$ and hence $x = \sqrt{2}$.

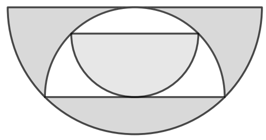
The area of a circle with radius r is πr^2. Therefore the area of a semicircle with radius r is $\frac{1}{2}\pi r^2$. Therefore the larger semicircle with radius 2 has area $\frac{1}{2}\pi 2^2 = 2\pi$, and the smaller semicircle with radius $\sqrt{2}$ has area $\frac{1}{2}\pi(\sqrt{2})^2 = \pi$.

It follows that the area of the larger semicircle that is shaded is $2\pi - \pi = \pi$. Hence the fraction of the larger semicircle that is shaded is $\dfrac{\pi}{2\pi}$, that is, $\dfrac{1}{2}$.

For investigation

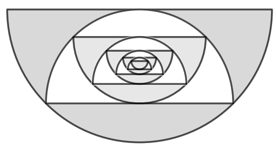
21.1 Prove that OPQ is a right-angled triangle in which $OP = QP$.

21.2 In the diagram on the right there are three semicircles. Both the largest and the smallest semicircle touch the middle semicircle at exactly three points.



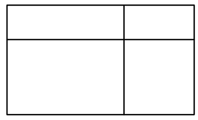
What fraction of the largest semicircle is shaded?

21.3 Suppose that the pattern shown in the previous problem is continued for ever, to produce an infinite sequence of semicircles each of which, after the first, touches the previous semicircle in exactly three places.



What fraction of the largest semicircle is shaded?

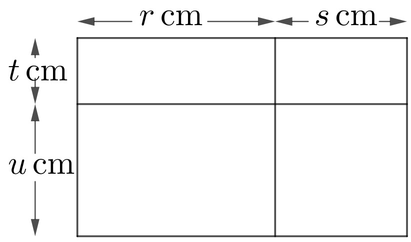
22. A rectangle with integer side-lengths is divided into four smaller rectangles, as shown. The perimeters of the largest and smallest of these smaller rectangles are 28 cm and 12 cm.
Which of the following is a possible area of the original rectangle?



A $90\,\text{cm}^2$ B $92\,\text{cm}^2$ C $94\,\text{cm}^2$ D $96\,\text{cm}^2$ E $98\,\text{cm}^2$

SOLUTION **D**

We let the side lengths of the four rectangles be as shown in the diagram.



We suppose that $r \geq s$ and $u \geq t$. It follows that the largest of the smaller rectangles has dimensions $r\,\text{cm} \times u\,\text{cm}$. The perimeter of this rectangle is made up of two edges with length r cm and two of length u cm. Hence its perimeter has length $(2r + 2u)$ cm. It follows that $2r + 2u = 28$. Therefore $r + u = 14$, and hence

$$u = 14 - r. \qquad (1)$$

Similarly, the smallest of the four smaller rectangles has dimensions $s\,\text{cm} \times t\,\text{cm}$. Therefore $2s + 2t = 12$. Hence $s + t = 6$ and therefore

$$t = 6 - s. \qquad (2)$$

The large rectangle has dimensions $(r + s)\,\text{cm} \times (t + u)\,\text{cm}$ and hence has area $(r + s) \times (t + u)\,\text{cm}^2$.

Now, by (1) and (2), $(r+s)\times(t+u) = (r+s)\times((6-s)+(14-r)) = (r+s)\times(20-(r+s)) = n(20-n)$, where $n = r + s$.

Therefore the area of the large rectangle, in square centimetres, is a positive integer of the form $n(20 - n)$, where n is a positive integer.

When $n = 8$, we have $n(20 - n) = 8 \times 12 = 96$. It may be checked (see Problem 22.1) that none of the integers 90, 92, 94 and 98 may be expressed in this form.

In the context of the IMC we may safely conclude that the correct option is $96\,\text{cm}^2$.

(A complete solution would need to show that it is possible to find positive integers r, s, t and u with $r + u = 14$, $s + t = 16$, $(r + s) \times (t + u) = 96$, $r \geq s$ and $u \geq t$. You are asked to do this in Problem 22.2.)

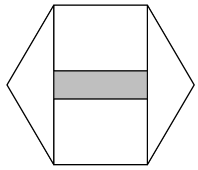
FOR INVESTIGATION

22.1 (a) Find all the positive integers that may be expressed in the form $n(20 - n)$, where n is a positive integer.

(b) Hence show that none of the integers 90, 92, 94 and 98 may be expressed in this form.

22.2 Find integers r, s , t and u such that $r + u = 14$, $s + t = 6$, $(r + s) \times (t + u) = 96$, $r \geq s$ and $u \geq t$.

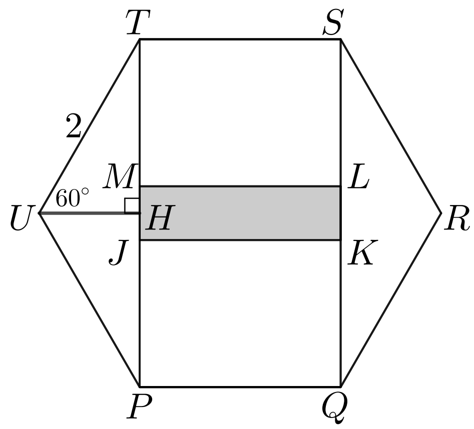
23. Two squares are drawn inside a regular hexagon with side-length 2, as shown. What is the area of the overlap of the two squares?



A 2 B $2 - \sqrt{3}$ C $4 - \sqrt{3}$ D $4 - 2\sqrt{3}$ E $8 - 4\sqrt{3}$

SOLUTION **E**

We let P, Q, R, S, T and U be the vertices of the hexagon, J and K be the other two vertices of the square with edge ST, and L and M be the other two vertices of the square with edge PQ.



Note that the two squares overlap in the quadrilateral $JKLM$. Because $STJK$ and $PQLM$ are squares, all the angles of this quadrilateral are right angles.

Therefore $JKLM$ is a rectangle.

It also follows that $JK = LM = PQ = 2$.

We let H be the midpoint of PT. It follows that the triangles TUH and PUH are congruent. Therefore $\angle THU$ is a right angle and $\angle HUT = 60°$. (You are asked to check all these facts in Problem 23.1.)

Because the hexagon has side-length 2, it follows that in the triangle TUH, we have $TH = \sqrt{3}$. (See Problem 23.2.)

Since $ST = 2$ and $STJK$ is a square, it follows that $TJ = 2$.

Therefore $JH = TJ - TH = 2 - \sqrt{3}$.

Similarly $HM = 2 - \sqrt{3}$. Hence $JM = JH + HM = 2(2 - \sqrt{3}) = 4 - 2\sqrt{3}$.

We can now deduce that the area of $JKLM$ is $2 \times (4 - 2\sqrt{3}) = 8 - 4\sqrt{3}$.

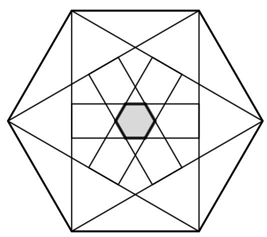
FOR INVESTIGATION

23.1 (a) Prove that the triangles TUH and PUH are congruent.

(b) Deduce that $\angle THU = 90°$ and $\angle HUT = 60°$.

23.2 Prove that $TH = \sqrt{3}$.

23.3 A regular hexagon has side-length 2. A square is drawn inside the hexagon on each edge of the hexagon. What is the area of the region, shaded in the diagram on the right, that is common to all six of these squares?



24. Pete's pies all cost an integer number of pounds. A cherry pie costs the same as two apple pies. A blueberry pie costs the same as two damson pies. A cherry pie and two damson pies cost the same as an apple pie and two blueberry pies. Paul buys one of each type of pie.
Which of the following could be the amount he spends?

A £16 B £18 C £20 D £22 E £24

Solution **B**

Let £a, £b, £c and £d be the cost of an apple pie, a blueberry pie, a cherry pie and a damson pie, respectively.

By what we are told in the question,

$$c = 2a,$$
$$b = 2d,$$

and

$$c + 2d = a + 2b.$$

Substituting from the first two of these equations into the third equation,

$$2a + 2d = a + 4d,$$

from which it follows that

$$a = 2d.$$

Hence

$$c = 2a = 4d.$$

Therefore the amount Paul spends buying one of each pie is, in pounds,

$$\begin{aligned} a + b + c + d &= 2d + 2d + 4d + d \\ &= 9d. \end{aligned}$$

Because d is an integer, it follows that the amount Paul spends is a multiple of £9. Therefore, of the given options, the only one that could be the amount that Paul spends is £18.

For investigation

24.1 After changes in the price of the ingredients Pete changed the prices of his pies.

He now charged £3.50 for a cherry pie. A blueberry pie now cost the same as three damson pies. The cost of an apple pie together with a blueberry pie was now the same as the cost of a cherry pie. A cherry pie and two damson pies now cost the same as two apple pies and one blueberry pie.

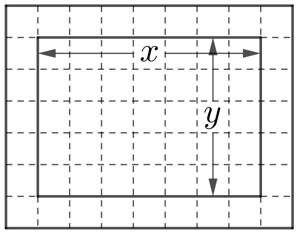
How much did it now cost Paul to buy one of each type of pie?

25. Alvita is planning a garden patio to be made from identical square paving stones laid out in a rectangle measuring x stones by y stones. She finds that when she adds a border of width one stone around the patio, the area of the border is equal to the original area of the patio.
How many possible values for x are there?

A 1 B 2 C 4 D 8 E 16

Solution **C**

The $x \times y$ rectangle contains xy stones. We see from the diagram that the border contains $2x + 2y + 4$ stones. Therefore, because the border has the same area as the rectangle,



$$2x + 2y + 4 = xy. \qquad (1)$$

Equation (1) may be rearranged to give $xy - 2x = 2y + 4$, and hence as

$$x(y - 2) = 2y + 4. \qquad (2)$$

If $y = 2$, then, by (2), $0 = 8$, which is impossible. Therefore $y \neq 2$. Thus $y - 2 \neq 0$, and therefore it follows from (2) that

$$\begin{aligned} x &= \frac{2y + 4}{y - 2} \\ &= 2 + \frac{8}{y - 2}. \end{aligned}$$

Since x is a positive integer, $y - 2$ is a factor of 8, and therefore $y - 2$ is ± 1, ± 2, ± 4 or ± 8. However, as y is also a positive integer, the only possible values for $y - 2$ are -1, 1, 2, 4 and 8.

When $y - 2 = -1$, we have $x = 2 - 8 = -6 < 0$ which is not possible.

Therefore $y - 2$ can only take the values, 1, 2, 4 and 8, giving $x = 2 + 8 = 10$, $x = 2 + 4 = 6$, $x = 2 + 2 = 4$, and $x = 2 + 1 = 3$, respectively.

Therefore there are 4 possible values for x.

For investigation

25.1 Find the values of y corresponding to the 4 possible values of x.

25.2 Check that $\dfrac{2y + 4}{y - 2} = 2 + \dfrac{8}{y - 2}$.

25.3 Suppose that when Alvita adds a border of width *two* stones around the patio, the area of the border is equal to the original area of the $x \times y$ patio. How many possible values for x are there in this case?

25.4 Suppose that when Alvita adds a border of width *three* stones around the patio, the area of the border is equal to the original area of the $x \times y$ patio. How many possible values for x are there in this case?